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Hamiltonicity in The Chained Cubic Tree Interconnection Network $CCT(h, n)$

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Abstract

Hamiltonicity plays a crucial role in the design of efficient interconnection networks, as it facilitates optimal routing of information between processors. The study of Hamiltonicity has inspired the development of several related concepts such as Hamiltonian laceability and Hamiltonian connectedness, fault tolerance and path covers etc., have lot of significance in computer networks as the presence of Hamiltonian path in network graphs is vital for addressing data communication challenges. A connected graph G is described as Hamiltonian$-t_{\text{odd}}(t_{\text{even}})-$laceable if for every pair of vertices x and y, there exists a Hamiltonian path such that the distance $d_s(x, y) = t$ where t is any odd (even) value satisfying $1 \leq t \leq Dm(G)$, with $Dm(G)$ representing the graph's diameter. Furthermore, G is Hamiltonian$-t-$connected when it satisfies Hamiltonian$-t-$laceable condition for every t in the range $1 \leq t \leq Dm(G)$. This article examines the Hamiltonian $t_{\text{odd}}-$laceability and Hamiltonian $t_{\text{even}}-$laceability properties of a chained cubic tree interconnection network, denoted as $CCT(h, n)$ specifically for $h = 1$, $n \geq 2$. The study's findings emphasize the distance specific Hamiltonian$-t-$connectedness of $CCT(h, n)$ characterized by $d_s(x, y) = t$, for $1 \leq t \leq Dm(G)$.

Keywords: Hamiltonicity; Hamiltonian laceability; connectedness; laceability number.

1 Introduction

Hamiltonian properties such as Hamiltonian laceability and Hamiltonian connectedness have paved the way for numerous innovations in the field of interconnection networks. Santiago Arguello et al. [12] have explored the Hamiltonicity in power graphs of a class of abelian groups. Shabbir et al. [13] have examined Hamiltonian connectedness in Toeplitz graphs. Liu et al. [9] have explored the sufficient conditions for Hamiltonicity of the graph and beyond. Inspired by these studies we delve into the properties of laceability and connectedness within an interconnection network, which are crucial for the efficient operation of modern computer systems. Hamiltonicity, characterized by the existence of Hamiltonian paths and Hamiltonian cycles is vital for ensuring network connectivity even in the event of partial network failures.

Assiri et al. [3] have explored fault tolerance in biswapped multiprocessor interconnection networks. Khan et al. [8] have investigated Hamiltonicity properties such as laceability and connectedness on planar graphs. Further, Hamiltonian properties of dragonfly network was studied by Wu et al. [17]. Zhong et al. [18] have explored Hamilton connectivity of line graphs with application to their detour index. Moreover, Shen et al. [14] have studied Hamilton connected property of Mycielski graphs and the laceability in the image graph of some classes of graphs was explored by Annapoorna and Murali [2].

A graph G' is termed as F-edge fault tolerant with respect to a graph G if, after removing any set of F edges from G', the resulting graph still contains G as a subgraph. The edge fault-tolerant Hamiltonicity to measure the performance of the Hamiltonian property in the faulty networks is proposed by Hsieh et al. [7]. Singh and Murali [15] establishes Hamiltonian laceability in the interleaver graph $C(2n, 1, 5)$ derived from the brick-product graph demonstrating strong path-traversal structure suitable for network applications. The work highlights how the graph's modular construction supports fault-tolerant routing by guaranteeing Hamiltonian paths between vertices in opposite partitions.

The study in [16] introduces and proves $t-$laceability properties in hierarchical hypercube networks, showing distance-restricted Hamiltonian path existence across different levels of the hierarchy. It strengthens the theoretical foundation for using hierarchical hypercubes in parallel computing by ensuring distance-specific path reliability and efficient communication. Among various interconnection networks, the hypercube is one of the most widely known due to its desirable properties. However, its limitations have prompted researchers to explore more efficient alternatives. In 2011, Abdullah et al. [1] proposed the chained-cubic tree (CCT) interconnection network, which demonstrated promising features such as low diameter and high connectivity. Despite these strengths, the Hamiltonicity properties of the CCT network remain relatively unexplored. Motivated by their work, we investigate the laceability properties of $CCT(h, n)$ for $h = 1$ and $n \geq 2$.

2 Preliminaries

An interconnection network is a structure that defines how multiple processing nodes or processors communicate with each other. The hypercube Q_n serves as an interconnection network, providing a structured and efficient method of nodes to exchange data. The graph Q_n consists of 2^n nodes where each node is uniquely identified by a binary string $a_{n-1}, a_{n-2}, a_{n-3}, \ldots, a_0$ of length n. In Q_n, two nodes are connected if and only if their binary representations differ in exactly one bit position. Hypercubes exhibit key properties of symmetry, regularity, and a bipartite nature, making them highly versatile in graph theory [16]. A bipartite graph $G(X, Y, E)$ is Hamiltonian laceable if there exists a $(x - y)$ Hamiltonian path between every pair of vertices in G such that $x \in X$ and $y \in Y$. Further a connected graph G is Hamiltonian$-t-$laceable (Hamiltonian$-t^*-$laceable) if there exists in G, a $(x - y)$ Hamiltonian path between every pair (atleast one of it's pair) of vertices with the property that $d_s(x, y) = t$ such that $1 \leq t \leq Dm(G)$. A connected graph G is Hamiltonian$-t-$connected if it is Hamiltonian$-t-$laceable for every t.

Fault tolerance refers to a system's ability to maintain proper functionality even some of it's components fails. Fault-tolerant computer systems are specifically designed with this principle in mind, ensuring they can continue to operate at an acceptable performance level despite the presence of faults. Let F_e represents the minimum set of edges not included in G. If P is a path in G, then $P \cup F_e$ forms a Hamiltonian path in G connecting u and v, where $|F_e|$ is refered to as the $t-$ laceability number, denoted by $\lambda_{(t)}$. Additionally, the edges in F_e are known as the fault tolerant$-t-$laceability edges of G with respect to u and v. [16].

Hamiltonian path or $H-$path in chained cubic tree networks $CCT(h, n)$ is highly significant in communication networks, parallel computing, robotics, circuit design, and other optimization problems. It provides an efficient way to visit all vertices exactly once, minimizing redundancy in data transmissions, optimizing resource usage, and improving traversal efficiency in practical systems and ensures balanced task distribution without revisiting nodes, which minimizes idle time and improves computational efficiency. The $CCT(h, n)$ network can model complex transportation systems where $H-$path help design delivery routes that visit each location (node) exactly once, reducing fuel costs and travel time. Within interconnections, failures of nodes or links are common and $H-$path help design alternative paths and these precomputed paths can serve as backup routes when primary paths fail, enhancing fault tolerance.

Qiao et al. [11] have studied the problem of embedding spanning disjoint paths in the enhanced hypercube networks with edge fault tolerance. Hamiltonian properties, particularly Hamiltonian connectedness and Hamiltonian laceability, play a crucial role in determining the reliability and communication efficiency of large-scale interconnection and data center networks. Further the problem of data harvest and information transmission and the importance to design the disjoint path cover in the logic graph of data center network has been investigated in [6] and path covers of bubble sort star graphs have been effectively examined in [5].

Azura et al. [4] explore Hamiltonian behavior within a modified chordal-ring network, a classical model used in parallel and distributed systems. Their analysis highlights how Hamiltonian structures contribute to efficient communication in interconnection networks. This perspective complements the network-oriented motivation of the present study, where Hamiltonian paths play a central role in describing reliable and distance-specific routing in $CCT(h, n)$. Malekpour and Bazigaran investigate Hamiltonian properties in graphs obtained from lattice-based constructions. Although their setting differs from interconnection networks, the emphasis on structural conditions that guarantee Hamiltonicity provides a useful theoretical parallel to our approach. Their results support the broader mathematical context in which distance-specific Hamiltonian laceability of $CCT(h, n)$ can be understood [10].

In modern data center topologies such as hypercubes, folded hypercubes, and butterfly networks, Hamiltonian-connected and laceable characteristics have been extensively analyzed to optimize message passing and minimize latency. By investigating these properties in newer structures like the chained cubic tree $CCT(h, n)$ networks, it becomes possible to evaluate their suitability for real-world deployment. The study of Hamiltonian laceability in $CCT(h, n)$ not only broadens the theoretical foundation of interconnection networks but also provides valuable insights into designing topologies that maintain strong connectivity even under link or node failures.

3 Chained Cubic Tree Network

A chained cubic tree $CCT(h,n)$ based on a tree T of height h as T_h and a hypercube Q_n of dimension n is obtained by replacing $2^{h+1}-1$ vertices of T_h by Q_n, $n \geq 2$ and connecting every vertex in the parent hypercube with the corresponding vertices in the children hypercubes. For sibling hypercubes at the same level, the hierarchical connections are used to connect the nodes that differ in the prefix bit. The parent - child vertical connections and horizontal sibling connections are defined as follows;

1. **PQ_n to C_LQ_n connections (parent to left child):**
   Connect the vertex x_{ik} to $x_{(i+1)k}$ where $x_{ik} \in PQ_n$ and $x_{(i+1)k} \in C_LQ_n$, for
   $$i = 1 \quad \text{and} \quad 1 \leq k \leq 2^n.$$
2. **PQ_n to C_RQ_n connections (parent to right child):**
   Connect the vertex x_{ik} to $x_{(i+2)k}$ where $x_{ik} \in PQ_n$ and $x_{(i+2)k} \in C_RQ_n$, for
   $$i = 1 \quad \text{and} \quad 1 \leq k \leq 2^n.$$
3. **C_LQ_n to C_RQ_n connections (sibling connections-left to right child):**
   Connect the vertex $x_{(i+1)k}$ to $x_{(i+2)j}$ where $x_{(i+1)k} \in C_LQ_n$ and $x_{(i+2)j} \in C_RQ_n$, for
   $$i = 1, \quad 1 \leq k \leq 2^n, \quad j = 2^n - l \quad \text{and} \quad 0 \leq l \leq 2^n - 1.$$

The construction of $CCT(1,3)$ is illustrated in Figure 1.

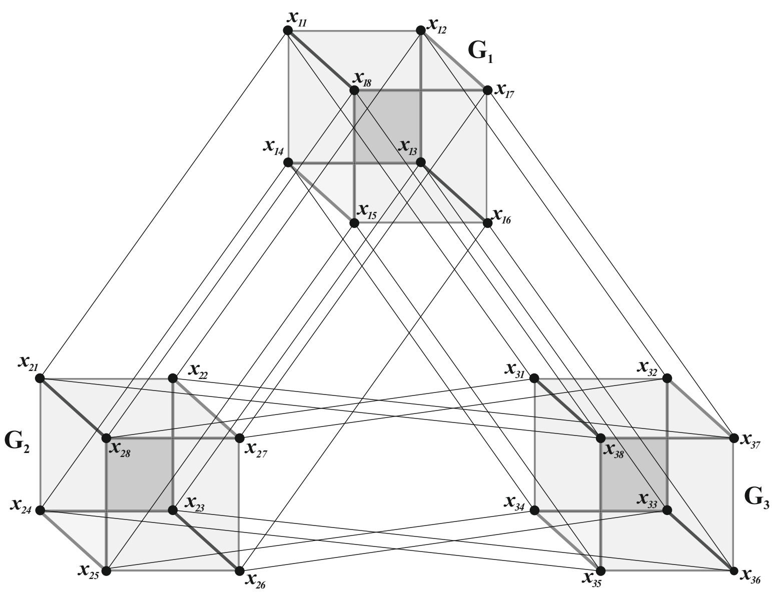

Figure 1: $CCT(1,3)$.

Let $G = CCT(h,n)$ for $h = 1$ and $n \geq 2$ be a chained cubic tree interconnection network whose vertex set is $|V(G)| = 2^n(2^{h+1}-1)$ and the edge set is $|E(G)| = 3 \times 2^n \left(\frac{n}{2}+1\right)$. Clearly the diameter of G is $2h+n-1$ [1]. Suppose $G_1=(V_1,E_1)$, $G_2=(V_2,E_2)$ and $G_3=(V_3,E_3)$ be three subgraphs of G which are disjoint and bipartite with equal number of vertices, then the one-one connection between G_1 and G_2, G_1 and G_3 and G_2 and G_3 is defined to be the set of edges as follows;

$$E_1^* = \left\{x_{ik} \text{ to } x_{(i+1)k} \text{ where } x_{ik}\in V_1 \text{ and } x_{(i+1)k}\in V_2,\ \text{for } i=1 \text{ and } 1\le k\le 2^n\right\},$$
$$E_2^* = \left\{x_{ik} \text{ to } x_{(i+2)k} \text{ where } x_{ik}\in V_1 \text{ and } x_{(i+2)k}\in V_3,\ \text{for } i=1 \text{ and } 1\le k\le 2^n\right\},$$
$$E_3^* = \Big\{x_{(i+1)k} \text{ to } x_{(i+2)j} \text{ where } x_{(i+1)k}\in C_LQ_n \text{ and } x_{(i+2)j}\in C_RQ_n \text{ for } i=1,\ \ 1\le k\le 2^n,$$
$$j=2^n-l \text{ and } 0\le l\le 2^n-1\Big\}.$$

Further, let us define a graph $G=\{V_1\cup V_2\cup V_3, E_m\cup E_m^*\}$ where $m=1,2,3$. Then the graph is said to be partitionable if $G=G_1\cup G_2\cup G_3$ [1].

3.1 Results and discussion

The existence of Hamiltonian$-t_{\text{odd}}(t_{\text{even}})-$laceability means that there exist long simple paths connecting each pair of vertices separated by odd or even distances, respectively, which allows for better communication primitives to be defined and enhances routing robustness under node or link failures. Since the chained-cubic tree $CCT(h,n)$ exposes a tree structure combined with clusters based on a hypercube, the Hamiltonian laceability results indicate that the hybrid structure retains much of the same path and cycle properties of the hypercube nested sub-components while growing in structural depth. Hence, the case $h=1$, $n\ge 2$ suggests the lowest structural height, indicating that even trees of low height exist with sufficient path guarantees that may be practically applicable to parallel architectures and hybrid optical/ electronic interconnection systems. Because the $CCT(h,n)$ integrates a tree hierarchy with hypercube-like clusters, the observed laceability confirms that it retains the favorable cycle and path characteristics of its hypercube components while maintaining hierarchical scalability.

Lemma 3.1. *The graph of $CCT(h,n)$ for $h=1$ and $n\ge 2$ is bipartite.*

Proof. Let $G=CCT(h,n)$ for $h=1$ and $n\ge 2$. G is bipartite if we partition the vertex set X of G into two subsets U and V such that no two vertices, either in U or in V are adjacent. Since the graph contains three subgraphs G_1,G_2 and G_3, it's vertices can be independently partitioned into two sets U_1, V_1 for G_1; U_2, V_2 for G_2 and U_3, V_3 for G_3. Let,

$$x_{ik}\in G_1,\qquad x_{(i+1)k}\in G_2\qquad\text{and}\qquad x_{(i+2)k}\in G_3,$$

be the set of vertices for $i=1$ and $1\le k\le 2^n$. We assign vertices x_{ik} of G_1 into two sets U_1, V_1 such that no two vertices within the same set are adjacent. This is always possible because G_1 is bipartite by itself. We extend the partition to G_2 and G_3. For each vertex $x_{(i+1)k}\in G_2$, we assign it to the opposite set of its corresponding vertex x_{ik} in G_1. That is, if a vertex x_{ik} in G_1 is in U_1, we assign its corresponding vertex $x_{(i+1)k}$ in G_2 to V_2 and if a vertex x_{ik} in G_1 is in V_1, then we assign its corresponding vertex $x_{(i+1)k}$ in G_2 to U_2.

Similarly for each vertex $x_{(i+2)k}$ in G_3, we assign it to the opposite set of its corresponding vertex $x_{(i+1)k}$ in G_2. Thus, if a vertex $x_{(i+1)k}$ in G_2 is in U_2, we assign its corresponding vertex $x_{(i+2)k}$ in G_3 to V_3 and if a vertex $x_{(i+1)k}$ in G_2 is in V_2, we assign its corresponding vertex $x_{(i+2)k}$ in G_3 to U_3. By construction G_1, G_2 and G_3 are bipartite as no edge connects vertices with in U or V. Further a vertex x_{ik} in G_1 (in U_1) connects to its corresponding vertex $x_{(i+1)k}$ in G_2 (in V_2) and a vertex $x_{(i+1)k}$ in G_2 (in U_2) connects its corresponding vertex $x_{(i+2)k}$ in G_3(in V_3), maintaining the bipartite structure of G. Thus no edges connects two vertices within U or within V across subgraphs. This concludes that $G = CCT(h, n)$ for $h = 1$ and $n \geq 2$ is bipartite. □

Theorem 3.1. *The graph of $CCT(h, n)$ for $h = 1$, $n \geq 2$ is Hamiltonian$-t_{odd}-$laceable with t being a positive integer satisfying $1 \leq t \leq Dm(G)$.*

Proof. Let $G = CCT(h, n)$ for $h = 1$ and $n \geq 2$. Then the vertex set of $G = CCT(h, n)$ is

$$|V(G)| = \{x_{ik} \cup x_{(i+1)k} \cup x_{(i+2)k}/i = 1, 1 \leq k \leq 2^n\},$$

and the edge set is

$$|E(G)| = \{E_m \cup E^*_m/1 \leq m \leq 3\}.$$

Clearly, the diameter of $G = CCT(h, n)$ is $2h + n - 1$. Due to G's bipartite nature, we partition the vertex set as,

$$V_1 = \left\{ v_x,\ \ 1 \leq x \leq \frac{2^n}{2} \right\},$$
$$V_2 = \left\{ v_y,\ \ 1 \leq y \leq \frac{2^n}{2} \right\}.$$

Let G be partitionable to G_p smaller subgraphs where $1 \leq p \leq 2^{h+1} - 1$, each of which is a Q_n, $n \geq 2$, having 2^n vertices. We divide the vertex set of G_1 to be $G_1(V) = \{\ g(V_1) \cup g(V_2)\}$ where,

$$g(V_1) = \left\{ g(v_{x_i})/1 \leq i \leq \frac{2^n}{2} \right\},$$
$$g(V_2) = \left\{ g(v_{y_i})/1 \leq i \leq \frac{2^n}{2} \right\}.$$

Similarly the vertex set of G_2 and G_3 be divided as $G_2(V) = \{\ g'(V_1) \cup g'(V_2)\}$ where,

$$g'(V_1) = \left\{ g'(v_{x_i})/1 \leq i \leq \frac{2^n}{2} \right\},$$
$$g'(V_2) = \left\{ g'(v_{y_i})/1 \leq i \leq \frac{2^n}{2} \right\},$$

and $G_3(V) = \{\ g''(V_1) \cup g''(V_2)\}$ where,

$$g''(V_1) = \left\{ g''(v_{x_i})/1 \leq i \leq \frac{2^n}{2} \right\},$$
$$g''(V_2) = \left\{ g''(v_{y_i})/1 \leq i \leq \frac{2^n}{2} \right\}.$$

Furthermore, G is $2-$colorable and it's vertices can be assigned two distinct colors such that no two adjacent vertices does not receive the same color. As a result the distance between any two vertices with different colors is odd, while the distance between vertices of same color is even. Let S and E be two vertices of different colors. Without loss of generality, assume S is colored red and E is colored black. A Hamiltonian path can be constructed in G that connects vertices S and E.

Case 1: S and E belong to the same subgraph of G satisfying,

$$d(S,E) = 2L-1, \qquad L \geq 1.$$

Let, the vertices S and $E \in G_1(= PQ_n)$ of G such that the distance between S and E represented by $d_s(S,E)$ is $(2L-1)$ for $L \geq 1$. We trace a path P_1 covering entire set of vertices of G_1 excluding the vertex $g(v_{x_1})$. The sequence of vertices in P_1 be

$$\left\{g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), g(v_{x_3}), \ldots, g\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right) = E\right\}.$$

Since G_1 is a parent hypercube and G_2 & G_3 are sibling hypercubes each of which is bipartite, the vertices of G_1 such as $g(v_{x_i})$ and $g(v_{y_i})$ are of different colors say red and black respectively. Their adjacent vertices $g'(v_{y_i})$ and $g'(v_{x_i})$ in G_2 are colored in black and red. Further the vertices adjacent to $g'(v_{y_i})$ and $g'(v_{x_i})$ in G_3 are $g''(v_{x_i})$ and $g''(v_{y_i})$ which are colored in red and black and so on. We find a path P_2 in G_2 from the vertex $g'(v_{y_1})$ to $g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)$ and P_3 in G_3 from the vertex $g''(v_{x_1})$ to $g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)$. The sequence of vertices in P_2 and in P_3 is

$$P_2 : \left\{g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), g'(v_{x_2}), \ldots, g'\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right\},$$

$$P_3 : \left\{g''(v_{x_1}), g''(v_{y_1}), g''(v_{x_2}), \ldots, g''\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with $2^h + 1$ edges,

$$\phi = [S = g(v_{x_1}), g'(v_{y_1})], \qquad \phi' = \left[g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right], \qquad \phi'' = [g''(v_{x_1}), g(v_{y_1})],$$

results in a Hamiltonian path, $P : \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi' \cup \phi''\}$, in G which connects the vertices S and E as depicted in Figure 2.

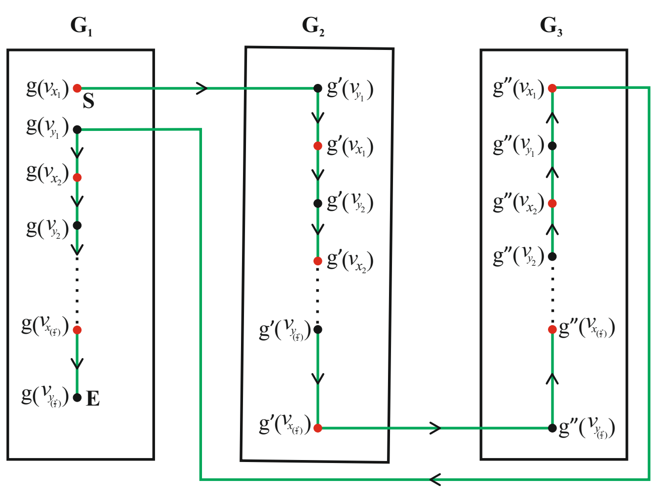

Figure 2: $H-$path between S and $E \in G_1(= PQ_n)$ of $CCT(1,n)$.

Case 2: S and E belong to different subgraphs of G such that,

$$d(S,E) = 2L - 1, \qquad L \geq 1.$$

Let, the vertices $S \in G_1(= PQ_n)$ and $E \in G_3(= C_RQ_n)$ of G such that the distance between S and E represented by $d_s(S,E)$ is $(2L-1)$ for $L \geq 1$. Then, we find paths P_1 of G_1, P_2 of G_2 and P_3 of G_3 whose sequences of vertices are

$$P_1 : \left\{ S = g(v_{x_1}), g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), g(v_{x_3}), \ldots, g\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right) \right\},$$

$$P_2 : \left\{ g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), g'(v_{x_2}), \ldots, g'\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right) \right\},$$

$$P_3 : \left\{ g''(v_{x_1}), g''(v_{y_1}), g''(v_{x_2}), \ldots, g''\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right) = E \right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with 2^h edges

$$\phi = \left[g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right) \right],$$
$$\phi' = \left[g'(v_{y_1}), g''(v_{x_1}) \right],$$

results in a Hamiltonian path $P : \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi'\}$ in G which connects the vertices S and E as depicted in Figure 3.

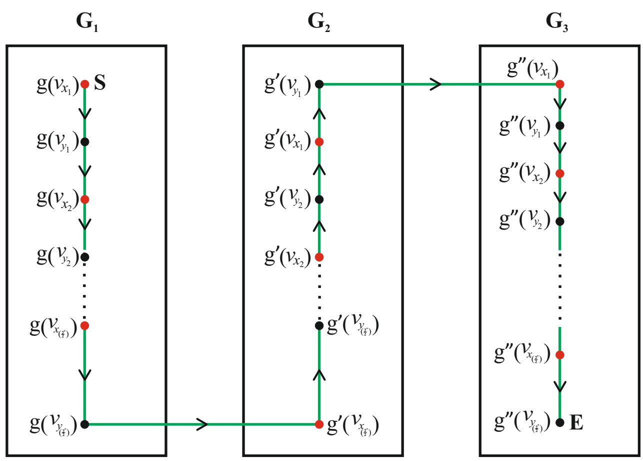

Figure 3: $H-$path between $S \in G_1(= PQ_n)$ and $E \in G_3(= C_RQ_n)$ of $CCT(1,n)$.

From the above two cases it is observed that there exists a Hamiltonian path between any two vertices belonging to different partite sets of G whose distance is odd. This proves that the graph of $CCT(h,n)$ for $h = 1$ and $n \geq 2$ is Hamiltonian$-t_{\text{odd}}-$laceable.

□

Theorem 3.2. *For $h = 1$ and $n \geq 2$, the graph $CCT(h, n)$ is Hamiltonian$-t_{even}-$laceable with fault tolerant$-t-$laceability edge $\lambda_{(t)} = 1$ where t is a positive integer satisfying $1 \leq t \leq Dm(G)$.*

Proof. Let $G = CCT(h, n)$ for $h = 1$ and $n \geq 2$. Let the cardinality of $G = CCT(h, n)$ be

$$|V(G)| = \{x_{ik} \cup x_{(i+1)k} \cup x_{(i+2)k}, \qquad i = 1, \qquad 1 \leq k \leq 2^n\},$$

and the order of G be

$$|E(G)| = \{E_m \cup E^*_m/1 \leq m \leq 3\}.$$

Clearly the diameter of $G = CCT(h, n)$ is $2h + n - 1$. Since hypercubes are bipartite, they are 2-colorable and the distance between any two vertices of identical colors is even.

Furthermore, we partition the vertices of G, $1 \leq p \leq 2^{h+1} - 1$ as,

$$V_1 \cup V_2 = \left\{v_x, 1 \leq x \leq \frac{2^n}{2}\right\} \cup V_2 = \left\{v_y, 1 \leq y \leq \frac{2^n}{2}\right\}.$$

Let, G be partitionable to G_p smaller subgraphs where $1 \leq p \leq 2^{h+1} - 1$, each of which is a Q_n, $n \geq 2$, having 2^n vertices, we divide the vertex set of G_1 to be $G_1(V) = \{\ g(V_1) \cup g(V_2)\}$, where

$$g(V_1) = \left\{g(v_{x_i})/1 \leq i \leq \frac{2^n}{2}\right\},$$
$$g(V_2) = \left\{g(v_{y_i})/1 \leq i \leq \frac{2^n}{2}\right\}.$$

Similarly, the vertex set of G_2 and G_3 be divided as $G_2(V) = \{\ g'(V_1) \cup g'(V_2)\}$, where

$$g'(V_1) = \left\{g'(v_{x_i})/1 \leq i \leq \frac{2^n}{2}\right\},$$
$$g'(V_2) = \left\{g'(v_{y_i})/1 \leq i \leq \frac{2^n}{2}\right\},$$

and $G_3(V) = \{\ g''(V_1) \cup g''(V_2)\}$ where

$$g''(V_1) = \left\{g''(v_{x_i})/1 \leq i \leq \frac{2^n}{2}\right\},$$
$$g''(V_2) = \left\{g''(v_{y_i})/1 \leq i \leq \frac{2^n}{2}\right\}.$$

Suppose S is a starting vertex and E is an ending vertex of a Hamiltonian path in G. With out loss of generality we take both S and E to be vertices of identical colors and trace the Hamiltonian path between S and E.

Case 1: S and E be red vertices belong to the same subgraph of G such that,

$$d(S, E) = 2L - 2, \qquad L \geq 2.$$

Let, the red vertices S and $E \in G_1(= PQ_n)$ of G such that the distance between S and E represented by $d(S, E)$ is $(2L - 2)$ for $L \geq 2$. Let $F_e = \{f_e, e = (f_1, f_2)/d(f_1, f_2) = 1\}$ be a faulty edge added to $G_1(= PQ_n)$, which makes the graph G to be non bipartite.

We trace a path P_1 of G_1 from the vertex $g(v_{x_1})$ to $g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right)$. The sequence of vertices in P_1 be,

$$P_1 : \left\{S = g(v_{x_1}), g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), \ldots, g\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right)\right\}.$$

Further, we find a path P_2 of G_2 and P_3 of G_3 whose sequence of vertices is

$$P_2 : \left\{g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), g'(v_{x_2}), \ldots, g'\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right\},$$

$$P_3 : \left\{g''(v_{x_1}), g''(v_{y_1}), g''(v_{x_2}), \ldots, g''\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with $2^h + 1$ edges,

$$\begin{aligned}
\phi &= \left[g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right], \\
\phi' &= [g'(v_{y_1}), g''(v_{x_1})], \\
\phi'' &= \left[g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right) = E\right] \cup F_e = \left\{f_1 = g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right),\right. \\
&\qquad\qquad\qquad\qquad \left. f_2 = g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right) / d(f_1, f_2) = 1 = \lambda_{(t)}\right\},
\end{aligned}$$

results in a Hamiltonian path $P : \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi' \cup \phi'' \cup F_e\}$ in G which connects the vertices S and E as depicted in Figure 4.

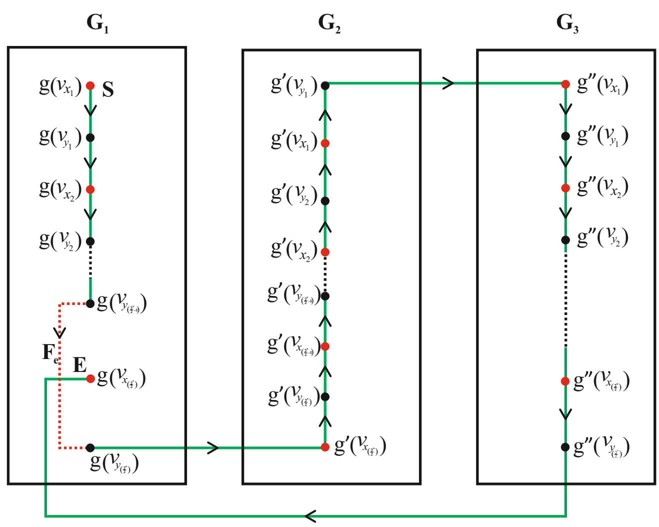

Figure 4: $H-$path between S and $E \in G_1(= PQ_n)$ of $CCT(1, n)$.

Case 2: S and E be red vertices belong to different subgraphs of G such that,

$$d(S, E) = 2L - 2, \qquad L \geq 2.$$

Let the red vertices $S \in G_1(= PQ_n)$ and $E \in G_2(= C_L Q_n)$ of G such that the distance between S and E represented by $d(S, E)$ is $(2L - 2)$ for $L \geq 2$.

Let, $F_e = \{f_e, e = (f_1, f_2)/d(f_1, f_2) = 1\}$ be a faulty edge added to $G_3(= C_R Q_n)$ makes the graph G to be non bipartite. We trace a path P_1 of G_1 from the vertex $g(v_{x_1})$ to $g\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)$. The sequence of vertices in P_1 be,

$$P_1 : \left\{S = g(v_{x_1}), g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), \ldots, g\left(v_{x_{\left[\frac{2n}{2}-1\right]}}\right), g\left(v_{y_{\left[\frac{2n}{2}-1\right]}}\right), g\left(v_{x_{\left[\frac{2n}{2}\right]}}\right), g\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)\right\}.$$

Further, we find a path P_2 of G_2 and P_3 of G_3 whose sequence of vertices is

$$P_2 : \left\{g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), \ldots, g'\left(v_{y_{\left[\frac{2n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2n}{2}\right]}}\right) = E\right\},$$

$$P_3 : \left\{g''(v_{x_1}), g''(v_{y_1}), g''(v_{x_2}), g''(v_{y_2}), \ldots, g''\left(v_{x_{\left[\frac{2n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2n}{2}-1\right]}}\right)\right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with 2^h edges,

$$\phi = [g''(v_{x_1}), g'(v_{y_1})],$$
$$\phi' = \left[g''\left(v_{x_{\left[\frac{2n}{2}\right]}}\right), g\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)\right],$$

and a faulty edge,

$$F_e = \left\{f_1 = g''\left(v_{y_{\left[\frac{2n}{2}-1\right]}}\right), f_2 = g''\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)/d(f_1, f_2) = 1 = \lambda_{(t)}\right\},$$

appending to an edge $\left[g''\left(v_{x_{\left[\frac{2n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)\right]$ results in a Hamiltonian path,

$$P : \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi' \cup F_e \cup \left[g''\left(v_{x_{\left[\frac{2n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2n}{2}\right]}}\right)\right],$$

in G which connects the vertices S and E as depicted in Figure 5.

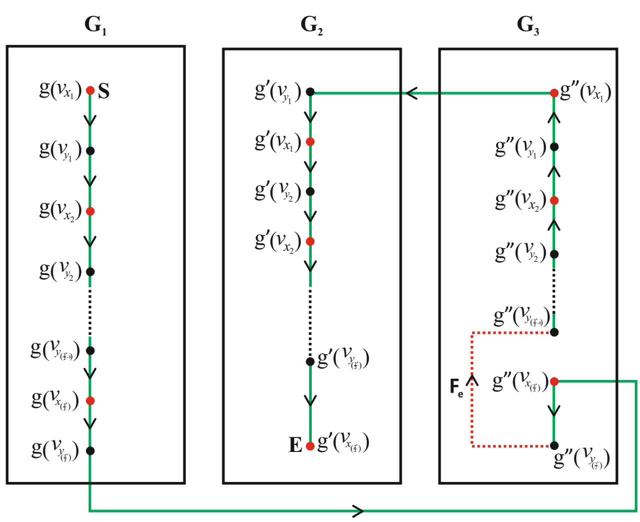

Figure 5: $H-$path between $S \in G_1(= PQ_n)$ and $E \in G_2(= C_L Q_n)$ of $CCT(1, n)$.

Case 3: S and E be black vertices belong to the same subgraph of G such that,

$$d(S,E) = 2L-2, \qquad L \geq 2.$$

Let the black vertices S and $E \in G_1(=PQ_n)$ of G such that the distance between S and E represented by $d(S,E)$ is $(2L-2)$ for $L \geq 2$. Let,

$$F_e = \{f_e, e = (f_1, f_2)/d(f_1,f_2) = 1\},$$

be a faulty edge added to $G_1(=PQ_n)$, which makes the graph G to be non bipartite. We trace a path P_1 of G_1 from the vertex $g(v_{y_1})$ to $g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)$. The sequence of vertices in P_1 be,

$$P_1 : \left\{S = g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), g(v_{x_3}), \ldots, g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

Further we find paths P_2 of G_2 and P_3 of G_3 whose sequence of vertices is

$$P_2 : \left\{g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), g'(v_{x_2}), \ldots, g'\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right\},$$

$$P_3 : \left\{g''(v_{x_1}), g''(v_{y_1}), g''(v_{x_2}), \ldots, g''\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with 2^h+1 edges,

$$\phi = [g(v_{x_1}), g'(v_{y_1})],$$

$$\phi' = \left[g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right],$$

$$\phi'' = \left[g''(v_{x_1}), g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right) = E\right] \cup F_e = \left\{f_1 = g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), f_2 = g(v_{x_1})/d(f_1,f_2) = 1 = \lambda_{(t)}\right\},$$

results in a Hamiltonian path $P : \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi' \cup \phi'' \cup F_e\}$ in G which connects the vertices S and E as depicted in Figure 6.

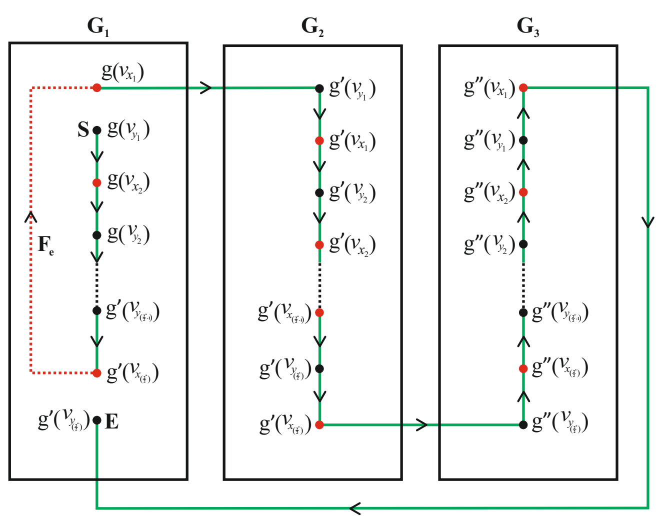

Figure 6: $H-$path between S and $E \in G_1(=PQ_n)$ of $CCT(1,n)$.

Case 4: S and E be two black vertices belong to different subgraphs of G such that,

$$d(S,E) = 2L-2, \qquad L \geq 2.$$

Let $S \in G_2(=C_LQ_n)$ and $E \in G_1(=PQ_n)$ and

$$F_e = \{f_e, e=(f_1,f_2)/d(f_1,f_2)=1\},$$

be a faulty edge added to $G_3(=C_RQ_n)$. This makes the graph G to be non bipartite.

We trace a path P_1 of G_1 from the vertex $g(v_{y_1})$ to $g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)$. The sequence of vertices in P_1 be

$$P_1: \left\{E = g(v_{y_1}), g(v_{x_2}), g(v_{y_2}), g(v_{x_3}), \ldots, g\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

Further, we find paths P_2 of G_2 and P_3 of G_3 whose sequence of vertices is

$$P_2: \left\{S = g'(v_{y_1}), g'(v_{x_1}), g'(v_{y_2}), g'(v_{x_2}), \ldots, g'\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g'\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right),\right.$$
$$\left. g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right)\right\},$$
$$P_3: \left\{g''(v_{x_2}), g''(v_{y_2}), g''(v_{x_3}), \ldots, g''\left(v_{x_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}-1\right]}}\right), g''\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right\}.$$

The union of paths P_1, P_2 and P_3 in conjunction with 2^h edges,

$$\phi = \left[g(v_{x_1}), g''\left(v_{y_{\left[\frac{2^n}{2}\right]}}\right)\right],$$
$$\phi' = \left[g'\left(v_{x_{\left[\frac{2^n}{2}\right]}}\right), g''(v_{y_1})\right],$$

with a faulty edge,

$$F_e = \left\{f_1 = g''(v_{x_1}), f_2 = g''(v_{x_2})/d(f_1,f_2) = 1 = \lambda_{(t)}\right\},$$

appending to an edge $[(g''(v_{x_1}), g''(v_{y_1})]$ results in a Hamiltonian path,

$$P: \{P_1 \cup P_2 \cup P_3 \cup \phi \cup \phi' \cup F_e \cup [(g''(v_{x_1}), g''(v_{y_1})],$$

in G which connects the vertices S and E as depicted in Figure 7.

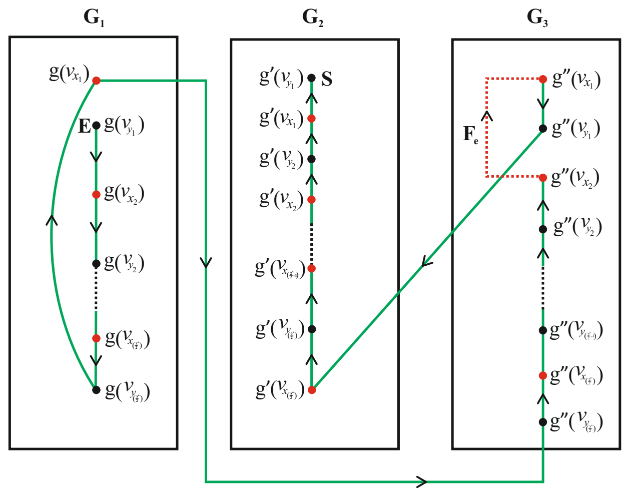

Figure 7: $H-$path between $S \in G_2 (= C_L Q_n)$ and $E \in G_1 (= PQ_n)$ of $CCT(1, n)$.

From the above cases it is observed that there exists a Hamiltonian path between any two vertices belonging to same partite sets of G whose distance is even. This proves that the graph of $CCT(h, n)$ for $h = 1$ and $n \geq 2$ is Hamiltonian$-t_{\text{even}}-$laceable. □

Theorem 3.3. *The graph of $CCT(h, n)$ for $h = 1$, $n \geq 2$ is Hamiltonian$-t-$connected.*

Proof. From Theorem 3.1, the graph of $CCT(h, n)$ for $h = 1$, $n \geq 2$ is Hamiltonian$-t_{\text{odd}}-$laceable where t is a positive integer such that $1 \leq t_{\text{odd}} \leq Dm(G)$. This implies that there exists a Hamiltonian path between every pair of vertices with the property that $d_s(x, y) = t_{\text{odd}}$ where $1 \leq t_{\text{odd}} \leq Dm(G)$. Further the Theorem 3.2 proves that the graph $CCT(h, n)$ for $h = 1$, $n \geq 2$ is Hamiltonian$-t_{\text{even}}-$laceable with the fault tolerant$-t-$laceability edge $\lambda_{(t)} = 1$ where t is a positive integer such that $1 \leq t_{\text{even}} \leq Dm(G)$. Thus G is Hamiltonian laceable for every t with $d_s(x, y) = t$ such that $1 \leq t \leq Dm(G)$ and therefore G Hamiltonian $-t-$ connected. □

3.2 Structural and performance features of $CCT(h, n)$ and related network topologies

The chained-cubic tree $CCT(h, n)$ topology is a hybrid model of networks that uses chains of hypercubes within the nodes of a tree, combining the advantages and reducing the disadvantages of tree networks and hypercube networks. The $CCT(h, n)$ model has better modular scalability and hierarchical growth compared to an $n-$dimensional hypercube, while still maintaining many of the good routing and embedding properties of hypercube networks. For many of the same reasons, $CCT(h, n)$ provide a useful topology for use in parallel and distributed computing systems where hierarchical organization, scalable node clustering, and efficient path or cycle embedding are important criteria for mapping algorithms and fault-tolerant communications.

Further the chained-cubic tree $CCT(h, n)$ network achieves an optimal balance between scalability and path efficiency, setting it apart from traditional interconnection topologies such as the hypercube, folded hypercube, and mesh. Unlike the hypercube, whose node degree grows exponentially with dimension, the $CCT(h, n)$ maintains a bounded node degree $\Delta = n + 1$ and a compact diameter $D = n+h-1$, simplifying implementation and expansion. Its dual scalability by adjusting either cube size n or tree height h allows for flexible system growth without exponential wiring complexity. The embedded hypercube clusters ensure the preservation of Hamiltonian and laceability properties, supporting efficient communication and fault-tolerant routing. Compared with mesh and tree-based networks, the $CCT(h, n)$ offers higher bisection width, improved fault resilience, and greater modularity, making it a robust and scalable solution for modern parallel and distributed computing environments.

The $CCT(h, n)$ network achieves a balanced integration of hierarchical scalability and path efficiency while preserving several favorable hypercube properties, demonstrating superior suitability for scalable parallel and distributed computing architectures.

Table 1 presents the comparative analysis of some important network such as $CCT(h, n)$, hypercube, folded hypercube, and 2-D mesh.

Table 1: Comparative analysis of $CCT(h, n)$, hypercube, folded hypercube, and 2-D mesh networks.

Property / Metric	**Hypercube** (Q_n)	**Folded hypercube** (FQ_n)	**2-D Mesh** $(M_{p\times q})$	**Chained cubic tree** $(CCT(h, n))$
Number of nodes (N)	2^n	2^n	$p \times q$	$(3 \times 2^n - 2)(2^h - 1)$
Node degree (Δ)	n	$n + 1$ (with one folded link)	$2, 3, 4$ (corner, edge, interior)	$n + 1$ (cube + one tree link)
Diameter (D)	n	$\lfloor n/2 \rfloor$	$(p - 1) + (q - 1)$	$n + h - 1$
Average distance (L)	$n/2$	$n/4$	$(p + q)/3$	$\approx (n/2) + (h/2)$
Bisection width (B)	2^{n-1}	2^{n-1}	$\min(p, q)$	$3 \times 2^{n-1}$ (for $h = 1$)
Fault tolerance	Up to $n - 1$ link or node failures without disconnection	Improved (extra folded links provide alternate paths)	Poor (a single link failure may isolate nodes)	Moderate to high; alternate tree links allow rerouting
Scalability	Exponential (adds 2^n nodes per dimension)	Exponential	Easily scalable by adding rows or columns	Hierarchical: extend by adding levels (h) or cube size (n)
Clustering structure	None (flat topology)	None (flat topology)	Grid-based	Hierarchical; local cubes connected through a tree structure
Hamiltonian cycle	Exists for all $n \geq 2$	Exists for all $n \geq 2$	Exists only for even p, q	Proven for $h = 1$, $n \geq 2$ (this work)
Hamiltonian laceability	Holds for all bipartite Q_n $(n \geq 2)$	Holds for even n in FQ_n	Rarely holds	Holds for $CCT(1, n)$, $n \geq 2$ (this work)
Embedding capability	Excellent (supports many subgraphs)	Excellent	Moderate	Strong (inherits hypercube embeddings within clusters)
Routing complexity	$O(n)$	$O(n)$	$O(p + q)$	$O(n + h)$
Topological type	Symmetric, regular	Symmetric, regular	Planar; non-regular at edges	Hierarchical hybrid (tree + cube)
Applications	Parallel computers, multiprocessors	Fault-tolerant parallel systems	VLSI and image-processing grids	Hierarchical multiprocessors, hybrid optical/electronic networks

4 Conclusions

This manuscript examines the properties of Hamiltonian$-t-$laceability and connectedness of the graph $CCT(h, n)$ for $h = 1$ and $n \geq 2$. Specifically, we have demonstrated that the graph of $CCT(h, n)$ for $h = 1$ and $n \geq 2$ is Hamiltonian$-t_{\text{odd}}-$laceable with $d_s(x, y) = t$ for every odd t in the range $1 \leq t \leq Dm(G)$ and Hamiltonian$-t_{\text{even}}-$laceable for all even t with laceability number $\lambda_{(t)} = 1$ satisfying $1 \leq t \leq Dm(G)$. Furthermore it would be of considerable interest to investigate if the graph of $CCT(h, n)$ for $h = 1$ and $n \geq 2$ is strongly Hamiltonian laceable.

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Conflicts of Interest The authors declare no conflict of interest.

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